NOTE ON THE TRANSFORMATION OF A DETERMINANT INTO ANY OTHER EQUIVALENT DETERMINANT.

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PROFESSOR Van Velzer's interesting note on the above subject in ANALYST, Vol. IX, pp. 116–118, has recalled to my mind a theorem to which I was led in dealing with the "Transformations connecting General Determinants with Continuants". (Trans. Roy. Soc. Edinb., XXX, pp. 5–14.) Taking determinants of the fourth order, the theorem is as follows:—

The first three elements of the last column (say) of the determinant $|\, a_1 \;\, b_2 \;\, c_3 \;\, d_4 \,|$ *may be replaced by any three magnitudes whatever,* a, β, γ, *provided the fourth element be changed into*

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 - a \\ b_1 & b_2 & b_3 & b_4 - \beta \\ c_1 & c_2 & c_3 & c_4 - \gamma \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \div |\, a_1 \;\, b_2 \;\, c_3 \,|.$$

For, calling the said fourth element x we are to have

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a \\ b_1 & b_2 & b_3 & \beta \\ c_1 & c_2 & c_3 & \gamma \\ d_1 & d_2 & d_3 & x \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 & a \\ b_1 & b_2 & b_3 & \beta \\ c_1 & c_2 & c_3 & \gamma \\ d_1 & d_2 & d_3 & 0 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ c_1 & c_2 & c_3 & 0 \\ d_1 & d_2 & d_3 & x \end{vmatrix}$$

$$\therefore \begin{vmatrix} a_1 & a_2 & a_3 & a_4 - a \\ b_1 & b_2 & b_3 & b_4 - \beta \\ c_1 & c_2 & c_3 & c_4 - \gamma \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = x \,|\, a_1 \;\, b_2 \;\, c_3 \,|$$

$$\text{and} \;\therefore\; x = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 - a \\ b_1 & b_2 & b_3 & b_4 - \beta \\ c_1 & c_2 & c_3 & c_4 - \gamma \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \div |\, a_1 \;\, b_2 \;\, c_3 \,|$$

as was to be shown.

The condition for the possibility of effecting the transformation is, as before, indicated by the occurrence of $|\, a_1 \;\, b_2 \;\, c_3 \,|$ as a *divisor* in the value of x.

Applying the theorem to the case of the transformation of

$$\begin{vmatrix} 0 & b & c \\ b & 1 & a \\ c & a & 1 \end{vmatrix} \quad \text{into} \quad \begin{vmatrix} 2abc & b & c \\ b & 1 & 0 \\ c & 0 & 1 \end{vmatrix}$$

we first change the column $c,\ a,\ 1$ into

$$c,\quad 0,\quad \begin{vmatrix} 0 & b & 0 \\ b & 1 & a \\ c & a & 1 \end{vmatrix} \div (-b^2),$$

i. e., into

$$c,\quad 0,\quad 1-(ac\div b);$$

and so on, exactly as Professor Van Velzer does.

This example fortunately is easy, and the process as applied to it appears to the best advantage. It is desirable however to see the shady side as well, and for this purpose I give the curious identity

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_5+b_2 & a_1+b_3 & a_2+b_4 & a_3+b_5 & a_4+b_1 \\ a_4+b_3 & a_5+b_4 & a_1+b_5 & a_2+b_1 & a_3+b_2 \\ a_3+b_4 & a_4+b_5 & a_5+b_1 & a_1+b_2 & a_2+b_3 \\ a_2+b_5 & a_3+b_1 & a_4+b_2 & a_5+b_3 & a_1+b_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_5-b_2 & a_1-b_3 & a_2-b_4 & a_3-b_5 & a_4-b_1 \\ a_4-b_3 & a_5-b_4 & a_1-b_5 & a_2-b_1 & a_3-b_2 \\ a_3-b_4 & a_4-b_5 & a_5-b_1 & a_1-b_2 & a_2-b_3 \\ a_2-b_5 & a_3-b_1 & a_4-b_2 & a_5-b_3 & a_1-b_4 \end{vmatrix}$$

which possesses considerable interest in the theory of alternants.

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INTEGRATION OF SOME GENERAL CLASSES OF TRIGONOMETRIC FUNCTIONS.

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$$\therefore \int \frac{dx}{(a+b\sec x)^n} = \int \frac{a\,dx}{(a+b\sec x)^{n+1}} - \frac{\tan x \sec x}{(a+b\sec x)^{n+1}} + (n+1)b$$

$$\times \int \frac{\sec^2 x\,dx}{(a+b\sec x)^{n+2}} + 2\int \frac{\sec^3 x\,dx}{(a+b\sec x)^{n+1}} - (n+1)b\int \frac{\sec^4 x\,dx}{(a+b\sec x)^{n+2}}.$$

Now

$$\frac{\sec^2 x}{(a+b\sec x)^{n+2}} = \frac{1}{b^2}\left[\frac{1}{(a+b\sec x^n} - \frac{2a}{(a+b\sec x)^{n+1}} + \frac{a^2}{(a+b\sec x)^{n+2}}\right]$$

$$\frac{\sec^3 x}{(a+b\sec x)^{n+1}} = \frac{1}{b^3}\left[\frac{1}{(a+b\sec x)^{n-2}} - \frac{3a}{(a+b\sec x)^{n-1}} + \frac{3a^2}{(a+b\sec x)^{n}} - \frac{a^3}{(a+b\sec x)^{n+1}}\right]$$

$$\frac{\sec^4 x}{(a+b\sec x)^{n+2}} = \frac{1}{b^4}\left[\frac{1}{(a+b\sec x)^{n-2}} - \frac{4a}{(a+b\sec x)^{n-1}} + \frac{6a^2}{(a+b\sec x)^{n}} - \frac{4a^3}{(a+b\sec x)^{n+1}} + \frac{a4}{(a+b\sec x)^{n+2}}\right].$$